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# ON THE ASYMPTOTIC BEHAVIOR OF ANY FUNDAMENTAL SOLUTION OF THE EQUATION OF ATMOSPHERIC DIFFUSION AND ON A PARTICULAR DIFFUSION PROBLEM

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# FOREWORD

This report is No. 8 of a series written for the Diffusion Project presently being conducted at Colorado Agricultural and Mechanical College for the Office of Naval Research under Contract N 9 onr 82401. The experimental phase of this project is being carried out in a wind-tunnel at the Fluid Mechanics Laboratory of the College. The project is under the general supervision of Dr. M. L. Albertson, Head of Fluid Mechanics Research of the Civil Engineering Department.

To Dr. M. L. Albertson, and to Dr. D. F. Peterson, Head of the Civil Engineering Department and Chief of the Civil Engineering Section of the Experiment Station, as well as to Professor T. H. Evans, Dean of the Engineering School and Chairman of the Engineering Division of the Experiment Station, the writer wants to express his appreciation for their kind interest in the present work.

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On the Asymptotic Behavior of any Fundamental Solution of the Equation of Atmospheric Diffusion and on a Particular Diffusion Problem[1]

by

Chia-Shun Yih

Abstract

In this paper, the asympotic behavior of any fundamental solution of the differential equation of atmospheric diffusion is studied. It is found that if the wind velocity and the diffusivity increases monotonically with height, then the "amplitude" and the spacing of the zeros of the fundamental solution will decrease asymptotically in certain definite ways. As an application a particular problem in atmospheric diffusion is solved at the end.

1. Introduction

If one neglects the longitudinal diffusivity in comparison with the vertical diffusivity, the equation of diffusion can be written as

$$U\frac{\partial c}{\partial x_1} = \frac{\partial}{\partial y_1}\left(K\frac{\partial c}{\partial y_1}\right)$$

where $c$ , $U$ , and $K$ are respectively the concentration, the wind velocity and the vertical diffusivity. $U$ and $K$ being functions of $y_1$ only, and $x_1$ and $y_1$ being measured respectively in the horizontal and the vertical directions.

With $c_o$ denoting the ambient concentration, and $h$ denoting a reference length, the last equation can be written in the dimensionless form:

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$$u\frac{\partial \phi}{\partial x} = \frac{\partial}{\partial y}\left(D\frac{\partial \phi}{\partial y}\right) \tag{1}$$

where

$$\phi = \frac{c - c_0}{c_0}, \quad x = \frac{x_1}{h}, \quad y = \frac{y_1}{h}, \quad u(y) = \frac{U}{U_0}, \quad D(y) = \frac{K}{U_0 h}$$

$U_0$ being a certain reference velocity. In atmospheric diffusion, $u$ and $D$ are usually assumed to be monotonically increasing functions of $y$.

To solve the differential system consisting of (1) and various boundary conditions, the method of separation of variables will be used. Assuming

$$\phi = X(x)\,Y(y) \tag{2}$$

and substituting in (1), one has

$$\frac{X'}{X} = \frac{(DY')'}{uY} = -\lambda^2 \tag{3}$$

where the primes denote differentiation $\lambda$ is a real constant which can be taken to be positive, and the negative sign on the right is necessitated by the boundary condition at $x = \infty$.

For convenience of discussion one writes (3) as

$$X' = -\lambda^2 X \tag{4}$$

$$(DY')' + \lambda^2 u Y = 0 \tag{5}$$

The solution of (4) being obviously

$$X = e^{-\lambda^2 x} \tag{6}$$

it is that of (5) which is of primary interest. In the following, one will endeavor to study the asymptotic properties of any non-trivial solution $Y(\lambda, y)$ of (5) where $\lambda$ will be assumed to be different from zero. As an application of the results obtained in the course of this study, a particular problem in atmospheric diffusion will be solved at the end.

## 2. The Asymptotic Behavior of $Y(\lambda, y)$

Multiplying (5) by D , and defining the new variable $\eta$ by

$$\eta = \int_0^y D^{-1} dy \tag{7}$$

one has

$$Y'' + \lambda^2 g Y = 0 \tag{8}$$

where

$$g(\eta(y)) = u(y) D(y) \tag{9}$$

In reality, D is different from zero at $y = 0$ , since molecular diffusivity is always present, and it must remain everywhere finite. Consequently the integral in (7) exists for any finite $y$ , but increases indefinitely as $y \to \infty$. Thus, D being a positive quantity, $\eta$ is a monotonically increasing function of $y$ mapping the interval $0 \leq y < \infty$ into $0 \leq \eta < \infty$ . Sometimes for convenience a simple functional form is assumed for D , for instance $D \sim y^n$ . But, n being usually less than 1, the interval $(0, \infty)$ is still mapped into itself by the transformation from $y$ to $\eta$. One will notice in addition that since u , D , and $\eta$ are all monotonically increasing functions of $y$ , $g(\eta)$ must be monotonically increasing function of $\eta$. With this in mind, one will consider (8) in the interval $0 \leq \eta < \infty$ .

Making the transformations

$$Y = G(\eta) F(\xi), \quad \xi = \xi(\eta) \tag{10}$$

and using primes to denote differentiation (with respect to $\eta$ and $\xi$ ), one has

$$Y' = G'F + GF'\xi'$$

$$Y'' = G''F + (2G'\xi' + G\xi'')F' + G\xi'\xi'F''$$

so that (8) becomes

$$G\xi'\xi'F'' + (2G'\xi' + G\xi'')F' + (\lambda g G + G'')F = 0 \qquad (11)$$

One demands that

$$2G'\xi' + G\xi'' = 0$$

integration of which gives

$$\xi' = G^{-2} \qquad (12)$$

$$\xi = \int_0^\eta G^{-2}\, d\eta \qquad (13)$$

as the simplest results. Thus

$$G\xi'\xi' = \frac{1}{G^3} \qquad (14)$$

and (11) becomes

$$F'' + (\lambda g G^4 + G'' G^3) = 0 \qquad (15)$$

Taking

$$G = g^{-\frac{1}{4}} \qquad (16)$$

one has

$$F'' + (\lambda + \tfrac{5}{16} g' g' g^{-3} - \tfrac{1}{4} g'' g^{-2})F = 0 \qquad (17)$$

From (13) and (16) one notices that

$$\xi = \int_0^\eta g^{\frac{1}{2}}\, d\eta \qquad (18)$$

so that $\xi$ is a monotonic increasing function of $\eta$, becoming $\infty$ as $\eta$ becomes $\infty$.

It will now be proved that the two terms involving $g$ in the parenthesis of (17) vanish asymptotically. One considers first the term $g'g'g^{-3}$. It is sufficient to show that

$$g' g^{-3/2} \to 0 \qquad \text{as} \quad \eta \to \infty$$

Letting
$$g'g^{-3/2} = s(\eta)$$

one has
$$-2g^{-1/2} = \int_{\infty}^{\eta} s(\eta)\,d\eta$$

where the lower limit is $\infty$ since $g^{-\frac{1}{2}} \to 0$ as $\eta \to 0$. Since $g$ is different from zero for all values of $\eta$ different from zero, $g^{-\frac{1}{2}}$ is finite for such values of $\eta$, so that the integral is convergent and $s(\eta)$ must vanish at $\infty$. Indeed, the same argument can be applied to prove that
$$g'g^{-m} \to 0 \qquad \text{as} \quad \eta \to \infty$$
if $m > 1$. Now, taking the term $g'g^{-2}$ and differentiating, one has
$$(g'g^{-2})' = g''g^{-2} - 2g'g'g^{-3}$$

The term on the left vanishes asymptotically since
$$g'g^{-2} \to 0 \qquad \text{as} \quad \eta \to \infty$$
and since $g$ is a smooth function. The second term on the right has just been shown to vanish asymptotically. Consequently,
$$g''g^{-2} \to 0 \qquad \text{as} \quad \eta \to \infty$$

Remembering that $0 \le \eta < \infty$ is mapped into $0 \le \xi < \infty$, instead of (17) one can study the equation
$$F'' + (\lambda^2 + q(\xi))\,F = 0 \tag{19}$$
where $q(\xi) \to 0$ as $\xi \to \infty$. It will be assumed that $q$ is monotonic asymptotically. After Courant and Hilbert (1931), one takes
$$F = \alpha \sin(\lambda\xi + \delta) \qquad F' = \alpha\lambda\cos(\lambda\xi + \delta) \tag{20}$$

where $\alpha$ and $\delta$ are functions of $\xi$, the asymptotic behaviors of which are to be investigated. Calculating $F''$ in two ways from (19) and the second equation of (20), one has

$$F'' = -(\lambda^2+q)\alpha \sin(\lambda\xi+\delta) = \lambda[\alpha' \cos(\lambda\xi+\delta) - \alpha(\lambda+\delta')\sin(\lambda\xi+\delta)]$$

so that

$$\tan(\lambda\xi+\delta) = \frac{\lambda\alpha'}{\alpha(\lambda\delta'-q)} \tag{21}$$

Obtaining $F'$ in two ways from (21), one has

$$F' = \alpha\lambda\cos(\lambda\xi+\delta) = \alpha'\sin(\lambda\xi+\delta) + \alpha(\lambda+\delta')\cos(\lambda\xi+\delta)$$

so that

$$\tan(\lambda\xi+\delta) = -\frac{\alpha\delta'}{\alpha'} \tag{22}$$

Multiplying (21) by (22), one has

$$\tan^2(\lambda\xi+\delta) = -\frac{\lambda\delta'}{\lambda\delta'-q} \tag{23}$$

from which it is easily seen that

$$\delta' = \frac{q}{\lambda}\sin^2(\lambda\xi+\delta) \tag{24}$$

Then (22) gives

$$\frac{\alpha'}{\alpha} = -\frac{q}{2\lambda}\sin 2(\lambda\xi+\delta) \tag{25}$$

which gives

$$\ln\alpha - \ln\alpha_\infty = \int_\beta^\xi \frac{\alpha'}{\alpha}\,d\xi \tag{26}$$

One now seeks to establish the convergence of the integral

$$\int_\beta^\infty \frac{\alpha'}{\alpha}\,d\xi = -\frac{1}{2\lambda}\int_\beta^\infty q\sin 2(\lambda\xi+\delta)\,d\xi \tag{27}$$

Putting

$$\nu = 2(\lambda\xi+\delta)$$

one can write the integral as

$$-\frac{1}{4}\int_{\nu(\beta)}^\infty \frac{q}{\lambda(\lambda+\delta')}\sin\nu\,d\nu = -\frac{1}{4}\int_{\nu(\beta)}^\infty \frac{q}{\lambda(\lambda+q\sin^2\frac{\nu}{2})}\sin\nu\,d\nu$$

Rembering that, with q vanishing monotonically for sufficiently large values of $\xi$ and with $\lambda$ being positive, the quantity

$$\frac{q}{\lambda(\lambda + q\sin^2\frac{\nu}{2})}$$

is unique in sign for large $\xi$ , but vanishes as $\xi \to \infty$, it is sufficient to show that

$$\int_{(N-1)\pi}^{N\pi} \frac{q}{\lambda(\lambda+q\sin^2\frac{\nu}{2})}\sin\nu\, d\nu > \int_{N\pi}^{(N+1)\pi} \frac{q}{\lambda(\lambda+q\sin^2\frac{\nu}{2})}\sin\nu\, d\nu$$

It is then sufficient to show that

$$\frac{q(N\pi - \Delta\nu)}{\lambda\left(\lambda+q(N\pi+\Delta\nu)\sin^2\frac{N\pi+\Delta\nu}{2}\right)} - \frac{q(N\pi+\Delta\nu)}{\lambda\left(\lambda+q(N\pi+\Delta\nu)\sin^2\frac{N\pi-\Delta\nu}{2}\right)} > 0$$

where $q\,(N\pi \pm \Delta\nu)$ are the values of q evaluated at $N\pi \pm \Delta\nu$ respectively. But, observing that

$$\sin^2\left(\frac{N\pi-\Delta\nu}{2}\right)-\sin^2\left(\frac{N\pi+\Delta\nu}{2}\right) = \cos(N\pi+\Delta\nu) - \cos(N\pi-\Delta\nu) = 0$$

the left side of the inequality is equal to

$$\frac{q(N\pi-\Delta\nu)-q(N\pi+\Delta\nu)}{\left(\lambda+q(N\pi-\Delta\nu)\sin^2\frac{N\pi-\Delta\nu}{2}\right)\left(\lambda+q(N\pi+\Delta\nu)\sin^2\frac{N\pi+\Delta\nu}{2}\right)}$$

which is greater than zero asymptotically since asymptotically q is monotonically decreasing. For q monotonically vanishing asymptotically one can therefore write

$$\ln\alpha_\infty = \ln\alpha_\beta + \int_\beta^\infty \frac{\alpha'}{\alpha}\, d\xi$$

Thus for such a function q the quantity $\alpha_\infty$ exists and the "amplitude" of a solution of (19) is asymptotically constant.

Going back to the original variable y and remembering (16) and (19), one concludes that the amplitude of Y decreases

[illegible]

[illegible] in [illegible] , [illegible]

[illegible]

[illegible] of $q$ [illegible]

[illegible]

$q$ is [illegible] ; [illegible]

[illegible] one can [illegible] write

[illegible]

Thus for [illegible] $q$ the quantity [illegible] , exists and the [illegible] (ii) [illegible].

Going back to the original variable $y$ and using (ii) and (iii), one concludes that the amplitude of $Y$ decreases

asymptotically as

$$\left[U(y)D(y)\right]^{-1/4}$$

One will next consider the "phase function" $\delta$. If

$$q = O\left(\frac{1}{x^M}\right) \qquad (M > 1)$$

or if $q$ is of even smaller order, then from (24) it can be seen that $\delta_\infty$ exists and the distribution of the zeros of $F$ will have the even spacing $\frac{\pi}{\lambda}$ with respect to $\xi$ asymptotically, in such a way that after a sufficiently large zero of $\xi$, all subsequent zeros can be approximately located from it by using the asymptotic spacing. If $q$ does not satisfy the requirement stated, then the accumulation of the difference between $\frac{\pi}{\lambda}$ and the actual spacing of zeros will become infinite, so that even if the spacing may be approaching $\frac{\pi}{\lambda}$ asymptotically, it is impossible to locate from this asymptotic spacing all the zeros subsequent to a sufficiently large one, without encountering grave errors after sufficiently many locations. What is true of the zeros of $F$ is of course also true of those of $F-A$, where $A$ is a fixed number.

## 3. The Solution of a Particular Problem in Atmospheric Diffusion

One considers the case where the vertical diffusivity and the wind velocity vary as power functions of height, and the ground is impervious to vapor. With the vapor concentration known to be a certain function of height at $x = 0$, it is proposed to calculate the vapor concentration for all positive values of $x$.

[illegible]

3. The [illegible]

[illegible]

values of x .

Writing

$$u = y^m \tag{28}$$

$$D = \frac{K_o}{U_o h} y^n \tag{29}$$

where $K_o$ is a reference diffusivity, (5) becomes

$$(y^n Y') + \mu^2 y^m Y = 0 \tag{30}$$

where

$$\mu^2 = \frac{\lambda^2 K_o}{U_o h} \tag{31}$$

According to (7), (16), (9), and (8)

$$\eta = \int_0^y y^{-n} dy = \frac{y^{1-n}}{1-n}$$

$$G = g^{-\frac{1}{4}} = y^{-\frac{m+n}{4}}$$

$$\xi = \int_0^\eta G^{-2} d\eta = \int_0^y y^{\frac{m-n}{2}} dy = \frac{2y^{\frac{m-n+2}{2}}}{m-n+2}$$

Thus, according to (10), the transformations

$$\xi = \frac{2y^{\frac{m-n+2}{2}}}{m-n+2} \qquad Y = y^{-\frac{m+n}{4}} F(\xi) \tag{32}$$

will carry (30) into

$$F'' + \left(\mu^2 + \frac{\frac{1}{4} - \sigma^2}{\xi^2}\right) F = 0 \tag{33}$$

where

$$\sigma = \frac{1-n}{m-n+2} \tag{34}$$

Since

$$g = y^{m+n} = (1-n)^{\frac{m+n}{1-n}} \eta^{\frac{m+n}{1-n}}$$

and

$$\frac{5}{16}\left(\frac{dg}{d\eta}\right)^2 g^{-3} - \frac{1}{4}\frac{d^2 g}{d\eta^2} g^{-2} = \left[\frac{1}{4} - \left(\frac{1-n}{m-n+2}\right)^2\right] \xi^{-2}$$

The fundamental solutions for $F$ (which for shortness will be called simply $F$ ) have the properties that $\measuredangle_\infty$ is constant and $\delta_\infty$ exists, so that $F$ is asymptotically periodic with a period $\frac{2\pi}{\lambda}$. As is well known, these fundamental solutions are

precisely $(\mu\xi)^{\frac{1}{2}} J_{\pm\sigma}(\mu\xi)$ , where $J_{\pm\sigma}(\mu\xi)$ are the Bessel functions of order $\pm\sigma$. Indeed, the principal asymptotic properties of the Bessel functions are deduced from those of F . The definitions of the Bessel functions are such that $\alpha_\infty$ for F is $-\sqrt{\frac{2}{\pi}}$ .

The asymptotic properties of F can be utilized, with the help of Dirichlet's integral theorem and (33), to furnish in a purely formal manner the formula due to McRobert (1931):

$$f(\xi) = \int_0^\infty t J_\beta(t\xi)\,dt \int_0^\infty s f(s) J_\beta(ts)\,ds \tag{35}$$

for $\beta > -\frac{1}{2}$ and a $f(\xi)$ vanishing sufficiently rapidly as $J \to \infty$. For a rigorous justification of the derivation, however, a few delicate points would have to be clarified. With this clarification, which will be rather burdensome, one will not be concerned at the moment. Instead, one will proceed with the solution of the proposed problem, which will be seen to depend on (35)

Since the ground is impervious to vapor, Y should satisfy the condition
$$\frac{dY}{dy} = 0 \quad \text{at } y = 0$$

The value of $\sigma$ being ordinarily positive and $J_{\pm\sigma}(\mu\xi)$ vary as $\xi^{\pm\sigma}$ near $\xi = 0$ , a simple calculation will show that the following solution of (30) should be used:

$$Y = y^{-\frac{m+n}{4}} F(\xi) = y^{-\frac{m+n}{4}} \xi^{\frac{1}{2}} F_{-\sigma}(\mu\xi) \sim \xi^{\sigma} J_{-\sigma}(\mu\xi)$$

Then the general solution of (1) is

$$\phi = \int_0^\infty \rho(\mu) e^{-\lambda^2 x} \xi^{\sigma} J_{-\sigma}(\mu\xi)\,d\mu$$

where $\lambda$ and $\mu$ are connected by (31). Let $\phi = f(\xi)$ at $x = 0$. The "density function" $\rho(\mu)$ should satisfy

$$\frac{f(\xi)}{\xi^{\sigma}} = \int_0^{\infty} \rho(\mu) J_{-\sigma}(\mu\xi)\, d\mu \tag{36}$$

But, $\sigma$ being ordinarily less than $\frac{1}{2}$, for such values of $\sigma$ one has, by (35):

$$\rho(\mu) = \mu \int_0^{\infty} s^{1-\sigma} f(s) J_{-\sigma}(\mu s)\, ds \tag{37}$$

so that the final solution is

$$\phi = \int_0^{\infty} \mu e^{-\frac{U_o h \mu^2}{K_o} x} \xi^{\sigma} J_{-\sigma}(\mu\xi)\, d\mu \int_0^{\infty} s^{1-\sigma} f(s) J_{-\sigma}(\mu s)\, ds \tag{38}$$

In order that the solution be valid, however $|f(\xi)|$ should be asymptotically of an order not higher than that of $\xi^{-p+\sigma}$, where $p > 2$.